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Distance-regular graphs which support a spin model are thin

Brian Curtin*

Department of Mathematics, Kyushu University, Hakozaki 6-10-1, Higashi-ku, Fukuoka 812-81, Japan

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Abstract

We show that any distance-regular graph whose Bose–Mesner algebra contains a spin model of a certain type is thin in the sense of Terwilliger.

Keywords: Spin model; Terwilliger algebra; Distance-regular graph; Association scheme

1. Introduction

A spin model W is a square matrix satisfying certain conditions which ensure that it yields an invariant of knots and links via a statistical mechanical construction of Jones [20]. Recently, Jaeger [18] proved that every symmetric spin model W is contained in the Bose–Mesner algebra of some association scheme. Immediately after the announcement of this result, Nomura [21] gave a canonical construction for a Bose–Mesner algebra $N(W)$ containing W for each symmetric spin model W. Nomura's construction was then generalized to include non-symmetric spin models by Jaeger, Matsumoto, and Nomura [19].

In this paper we investigate the situation $W \in M \subseteq N(W)$, where W denotes a spin model and M is the Bose–Mesner algebra of a distance-regular graph Γ. More precisely, we study the Terwilliger algebra of Γ in this case.

Let $\Gamma = (X, E)$ denote a distance-regular graph with diameter d, and fix a vertex x of Γ. The *Terwilliger algebra* $T = T(x)$ is the subalgebra of $\mathrm{Mat}_X(\boldsymbol{C})$ generated by A, $E_0^*, E_1^*, \ldots, E_d^*$, where A denotes the adjacency matrix for Γ and E_i^* denotes the projection onto the ith subconstituent of Γ with respect to x. An irreducible T-module W is said to be *thin* whenever $\dim E_i^* W \leqslant 1$ $(0 \leqslant i \leqslant d)$. Γ is said to be *thin with respect to x* whenever every $T(x)$ module is thin, and Γ is said to be

* E-mail: curtin@math.kyushu-u.ac.jp.

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thin whenever Γ is thin with respect to every vertex. Every distance-regular graph with diameter at most 2 is thin, but distance-regular graphs with diameter at least 3 need not be thin.

Our main result is the following theorem.

Theorem. *Let $\Gamma=(X,R)$ denote a distance-regular graph with diameter $d\geqslant 3$, and let M denote the Bose–Mesner algebra of Γ. Let W denote a spin model, and assume that $W\in M\subseteq N(W)$. Write $W=\sum_{i=0}^{d} t_iA_i$, where A_i is the ith distance-matrix of Γ, and assume that the scalars $t_0,t_1,\ldots,t_d$ satisfy $t_i\notin\{t_0,-t_0\}$ $(1\leqslant i\leqslant d)$. Then Γ is thin.*

2. Background

In this section we briefly review some background material. We begin by recalling some facts about commutative association schemes. For more details see [4] or [7].

Let X be a finite, non-empty set, and let $\mathrm{Mat}_X(\boldsymbol{C})$ denote the $\boldsymbol{C}$-algebra of matrices with entries in $\boldsymbol{C}$ whose rows and columns are indexed by X. For all $A\in\mathrm{Mat}_X(\boldsymbol{C})$ and for all $a,\, b\in X$, we write $A(a,b)$ to denote the (a,b)-entry of A. Let $V=\boldsymbol{C}^{|X|}$ denote the column vector space indexed by X. Observe that $\mathrm{Mat}_X(\boldsymbol{C})$ acts on V by left multiplication. For each $y\in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and zeros everywhere else. We endow V with the Hermitian inner product $\langle,\rangle$ defined by $\langle u,v\rangle=u^t\bar{v}$.

By a *commutative association scheme* (or simply *scheme* hereafter) we mean a pair $\mathscr{X}=(X,\{A_i\}_{i=0,1,\ldots,d})$, where X is a finite, non-empty set, and where $A_0,A_1,\ldots,A_d\in\mathrm{Mat}_X(\boldsymbol{C})$ are non-zero (0,1) matrices satisfying the following conditions: (i) $\sum_{i=0}^{d}A_i=J$ (the all ones matrix), (ii) $A_0=I$ (the identity matrix), (iii) for all i $(0\leqslant i\leqslant d)$ there exists an i' $(0\leqslant i'\leqslant d)$ such that $A_i^t=A_{i'}$, and (iv) for all h, i, and j $(0\leqslant h,i,j\leqslant d)$ there exists a scalar p_{ij}^h such that $A_iA_j=A_jA_i=\sum_{h=0}^{d}p_{ij}^hA_h$. A_i is called the ith *associate matrix* of $\mathscr{X}$.

Let $\mathscr{X}=(X,\{A_i\}_{i=0,1,\ldots,d})$ be a scheme. From (i)–(iv) we see that $A_0,A_1,\ldots,A_d$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\boldsymbol{C})$. We refer to M as the *Bose–Mesner algebra* of $\mathscr{X}$. By [4], M has a basis $E_0,E_1,\ldots,E_d$ satisfying: (i) $\sum_{i=0}^{d}E_i=I$, (ii) $E_0=|X|^{-1}J$, (iii) for all i $(0\leqslant i\leqslant d)$ there exists an $\hat{\imath}$ $(0\leqslant\hat{\imath}\leqslant d)$ such that $E_i^t=\bar{E}_i=E_{\hat{\imath}}$, and (iv) $E_iE_j=\delta_{ij}E_i$ $(0\leqslant i,j\leqslant d)$. We refer to $E_0,E_1,\ldots,E_d$ as the *primitive idempotents* of $\mathscr{X}$.

Observe that M is closed under entry-wise multiplication, $\circ$. By the *Hadamard algebra* on M we mean the vector space M together with $\circ$. Observe that the A_i are the primitive idempotents of the Hadamard algebra.

By a *duality* of M we mean a vector space isomorphism Ψ mapping M onto itself satisfying $\Psi(AB)=\Psi(A)\circ\Psi(B)$, $\Psi(A\circ B)=|X|^{-1}\Psi(A)\Psi(B)$, and $\Psi(\Psi(A))=|X|A^t$ for all $A,\, B\in M$. Whenever a duality of M exists, we say that M is *self-dual*.

Lemma 2.1. *Let $\mathscr{X}=(X,\{A_i\}_{i=0,1,\ldots,d})$ denote a scheme, and let M denote the Bose–Mesner algebra of $\mathscr{X}$. Suppose there exists a duality Ψ of M. Then there exists an ordering $E_0,E_1,\ldots,E_d$ of the primitive idempotents such that $\Psi(E_i)=A_i$ $(0\leqslant i\leqslant d)$. We refer to this ordering as the* standard ordering of the primitive idempotents with respect to Ψ.

Proof. Observe that for all i,j $(0\leqslant i,j\leqslant d)$, $\Psi(E_i)\circ\Psi(E_j)=\Psi(E_iE_j)=\delta_{ij}\Psi(E_i)$, so $\Psi(E_0),\Psi(E_1),\ldots,\Psi(E_d)$ form a basis of primitive idempotents for the Hadamard algebra on M. It follows that for an appropriate indexing $\Psi(E_i)=A_i$ $(0\leqslant i\leqslant d)$. □

Lemma 2.2. *Let $\mathscr{X}=(X,\{A_i\}_{i=0,1,\ldots,d})$ denote a scheme, and let M denote the Bose–Mesner algebra of $\mathscr{X}$. Suppose there exists a duality Ψ of M. Let $E_0,E_1,\ldots,E_d$ denote the primitive idempotents of $\mathscr{X}$ in the standard order with respect to Ψ.*
(i) $\Psi(A_i)=|X|E_{\hat{i}}$ $(0\leqslant i\leqslant d)$.
(ii) $\hat{i}=i'$ $(0\leqslant i\leqslant d)$.

Proof. (i) By definition $\Psi(\Psi(E_i))=|X|E_{\hat{i}}$. We have seen that $\Psi(E_i)=A_i$ by Lemma 2.1, so the result follows.
(ii) Observe that $|X|A_{i'}=\Psi(\Psi(A_i))=|X|\Psi(E_{\hat{i}})=|X|A_{\hat{i}}$. □

We now briefly review the connection between (not necessarily symmetric) spin models and commutative association schemes. For more information concerning this material see [19,21,2]. Let X be a finite, non-empty set. A *spin model on X* is a matrix $W\in\mathrm{Mat}_X(\boldsymbol{C})$ that has all entries non-zero, and which satisfies the following equations for all $a,b,c\in X$:

$$\sum_{x\in X}W(x,b)W(x,c)^{-1}=|X|\delta_{bc},$$

$$\sum_{x\in X}W(x,a)W(x,b)W(x,c)^{-1}=LW(a,b)W(c,b)^{-1}W(a,c)^{-1}$$

for some $L\in\boldsymbol{R}$ such that $L^2=|X|$. A spin model is said to be *symmetric* when it is a symmetric matrix. Each spin model gives rise to a scheme via the following construction of Nomura. For all $b,c\in X$, define $Y_{bc}\in V$ to have x coordinate $Y_{bc}(x)=W(x,b)/W(x,c)$ for all $x\in X$. Define $N(W)$ to be the set of all matrices $A\in\mathrm{Mat}_X(\boldsymbol{C})$ such that for all $b,c\in X$, the vector Y_{bc} is an eigenvector of A. By [19], $W\in N(W)$ and $N(W)$ is the Bose–Mesner algebra of some scheme.

Let W denote a spin model on X. For $A\in N(W)$, let $\Psi(A)\in\mathrm{Mat}_X(\boldsymbol{C})$ be defined by $AY_{bc}=(\Psi(A))(b,c)Y_{bc}$ for all $b,c\in X$. By [19], Ψ is a duality of $N(W)$, and the map Ψ can be expressed by

$$\Psi(A)=\alpha^{-1}W\circ(W''(W'\circ A))\quad(A\in N(W)),\tag{1}$$

where W' is defined by $W'(x,y)=W(y,x)^{-1}$ $(x,y\in X)$, and where α is the scalar given by $W'J=L\alpha J$. Observe that $\Psi(W)=LW''$ and $\Psi(W')=LW$.

Definition 2.3. Let $\mathscr{X}=(X,\{A_i\}_{i=0,1,\dots,d})$ denote a scheme, let M denote the Bose–Mesner algebra of $\mathscr{X}$, and let W be a spin model on X. Suppose $W\in M$. By the *weights* of W (with respect to $\mathscr{X}$) we mean the complex scalars $t_0,t_1,\dots,t_d$ such that $W=\sum_{i=0}^{d} t_iA_i$. The scheme $\mathscr{X}$ is said to *support* the spin model W whenever $W\in M\subseteq N(W)$.

An example is discussed in [19] of a scheme which supports a spin model W and whose Bose–Mesner algebra is properly contained in $N(W)$. Our aim is to study the distance-regular graphs which support a spin model. Before proceeding to this case, let us try to understand the notion of support better. In particular, we wish to stress that the assumption $M\subseteq N(W)$ is essential and independent of the assumption that $W\in M$. The notion of support is related to that of fusion schemes. Let $\mathscr{Y}=(X,\{B_i\}_{i=0,1,\dots,D})$ denote a scheme. By a *fusion scheme* of $\mathscr{Y}$ (or its Bose–Mesner algebra), we mean a scheme $\mathscr{X}=(X,\{A_i\}_{i=0,1,\dots,d})$ such that each A_i is the sum of a subset of $\{B_i\}_{i=0,1,\dots,D}$. For more on fusion schemes see, for example [6].

Lemma 2.4. *Let $\mathscr{X}=(X,\{A_i\}_{i=0,1,\dots,d})$ denote a scheme, let M denote the Bose–Mesner algebra of $\mathscr{X}$, and let W denote a spin model on X. Then $\mathscr{X}$ supports W if and only if $W\in M$ and $\mathscr{X}$ is a fusion scheme of $N(W)$.*

Proof. First suppose $\mathscr{X}$ supports $\mathscr{W}$. Then $W\in M$ and $A_i\in N(W)$ by definition. Thus each A_i is a linear combination of the associate matrices for $N(W)$. Since the associate matrices are zero-one matrices with no common non-zero entries, it must be the case that the coefficients of this linear combination are zero and one, and this implies that $\mathscr{X}$ is a fusion scheme of the scheme of $N(W)$. Conversely, suppose that $\mathscr{X}$ is a fusion scheme of $N(W)$. Then certainly each $A_i\in N(W)$, so $M\subseteq N(W)$. Thus $\mathscr{X}$ supports W. □

Schemes which support a spin model inherit a duality from $N(W)$.

Lemma 2.5. *Let $\mathscr{X}=(X,\{A_i\}_{i=0,1,\dots,d})$ denote a scheme, and let M denote the Bose–Mesner algebra of $\mathscr{X}$. Let W denote a spin model on X, assume that $W\in M$, and let $t_0,t_1,\dots,t_d$ denote the weights of W with respect to $\mathscr{X}$. Let Ψ denote the duality of $N(W)$ given by* (1).

(i) $W'=\sum_{i=0}^{d} t_{i'}^{-1}A_i$. *In particular, $W'\in M$.*

(ii) *Suppose $\mathscr{X}$ supports W. Then the restriction of Ψ to M is a duality of M. In particular, $\mathscr{X}$ is self-dual.*

Proof. (i) Clear since $W\circ(W')^t=J$.

(ii) For all $A\in M$, $\Psi(A)\in M$ by (1) since M is closed under transposition, Hadamard product, and matrix product. Now the restriction $\Psi|_M$ is a duality of M in light of the remarks preceding Lemma 2.1. □

Definition 2.6. Let $\mathscr{X}=(X,\{A_i\}_{i=0,1,\dots,d})$ denote a scheme, and let M denote the Bose–Mesner algebra of $\mathscr{X}$. Let W denote a spin model on X, and assume that $\mathscr{X}$ supports W.

An ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of $\mathscr{X}$ is said to be *standard with respect to* W whenever it is standard with respect to $\Psi|_M$, where Ψ is the duality of $N(W)$ given by (1).

Lemma 2.7. *Let $\mathscr{X} = (X, \{A_i\}_{i=0,1,\ldots,d})$ denote a scheme, and let M denote the Bose–Mesner algebra of $\mathscr{X}$. Let W denote a spin model on X. Suppose $\mathscr{X}$ supports W. Let $E_0, E_1, \ldots, E_d$ denote the primitive idempotents of $\mathscr{X}$ in the standard order with respect to W. Then*

$$W = L\sum_{i=0}^{d} t_i^{-1} E_i, \quad W' = L\sum_{i=0}^{d} t_i E_i.$$

Proof. We have seen that $\Psi(W') = LW$ by (1). Also, applying Lemma 2.2, we see that $\Psi(W') = \sum_{i=0}^{d} t_i^{-1}\Psi(A_i') = L^2\sum_{i=0}^{d} t_i^{-1}E_i$. It follows that $W = L\sum_{i=0}^{d} t_i^{-1}E_i$. Now, since $WW' = |X|I$, $W' = L\sum_{i=0}^{d} t_iE_i$. $\square$

Lemma 2.8. *Let $\chi = (X, \{A_i\}_{i=0,1,\ldots,d})$ denote a scheme, and let W denote a spin model on X. Let M denote the Bose–Mesner algebra of χ, assume $W \in M$, and let $t_0, t_1, \ldots, t_d$ denote the weights of W with respect to χ. Then*

(i) *$t_0, t_1, \ldots, t_d$ are distinct if and only if W generates M.*

(ii) *If these equivalent conditions hold, then $M \subseteq N(W)$, and χ supports W.*

Proof. Clearly a matrix $Z = \sum_{i=0}^{d} z_iE_i$ generates M if and only if the z_i are distinct. By Lemma 2.7, $W = L\sum_{i=0}^{d} t_i^{-1}E_i$, and (i) follows. If W generates M, then clearly $M \subseteq N(W)$, so χ supports W. $\square$

Although the algebra generated by a spin model W is a natural object to consider, it is not always a Bose–Mesner algebra, as shown by examples given in [3]. See also [17] for further discussion of this point. However, it is not necessary that a spin model generates the Bose–Mesner algebra of any scheme which supports it.

We now review some facts about distance-regular graphs and the Q-polynomial property. See [4] or [7] for more details. Let $\Gamma = (X, R)$ denote a finite, connected, undirected graph with no loops or multiple edges, let ∂ denote the shortest-path distance function of Γ, and let d denote the diameter of Γ. Γ is said to be *distance-regular* whenever for all integers i $(0 \leqslant i \leqslant d)$ and for all $x, y \in X$ with $\partial(x, y) = i$, the numbers $c_i = |\{z \in X \mid \partial(x,z) = i-1,\ \partial(y,z) = 1\}|$, $a_i = |\{z \in X \mid \partial(x,z) = i,\ \partial(y,z) = 1\}|$, and $b_i = |\{z \in X \mid \partial(x,z) = i+1,\ \partial(y,z) = 1\}|$ are independent of x and y. The constants c_i, a_i, and b_i are known as the *intersection numbers* of Γ. For $x \in X$, we write $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$ $(0 \leqslant i \leqslant d)$.

For each integer i $(0 \leqslant i \leqslant d)$, let A_i denote the matrix in $\mathrm{Mat}_X(\boldsymbol{C})$ with (x, y)-entry $A_i(x, y) = 1$ if $\partial(x, y) = i$ and 0 otherwise. The matrix A_i is called the ith *distance matrix* of Γ. The matrix $A = A_1$ is just the *adjacency matrix* of Γ. It is well known that $(X, \{A_i\}_{i=0,1,\ldots,d})$ is an association scheme. We use Γ to refer to both the graph and to the association scheme arising from the graph. Observe that every matrix in the Bose–Mesner algebra of a distance-regular graph is symmetric.

Observe that $A=\sum_{i=0}^{d}\theta_i E_i$ for some scalars $\theta_0,\theta_1,\ldots,\theta_d$. The scalar θ_i is known as the *eigenvalue* of Γ associated with E_i. It is well known that the eigenvalues of Γ are distinct. For any primitive idempotent E, there exist scalars $\theta_0^*,\theta_1^*,\ldots,\theta_d^*$ such that $E=|X|^{-1}\sum_{i=0}^{d}\theta_i^* A_i$. The scalars $\theta_0^*,\theta_1^*,\ldots,\theta_d^*$ are called the *dual eigenvalues of Γ associated with E*. Observe that for any $x,y\in X$, $\langle E\hat{x},E\hat{y}\rangle = |X|^{-1}\theta_h^*$, where $h=\partial(x,y)$.

An ordering $E_0,E_1,\ldots,E_d$ of the primitive idempotents is said to be a *Q-polynomial ordering* whenever for all distinct integers i,j $(0\leqslant i,j\leqslant d)$, $E_1(E_i\circ E_j)\neq 0$ if and only if $|i-j|=1$. Let E denote a primitive idempotent. Then Γ is said to be *Q-polynomial with respect to E* when there exists a Q-polynomial ordering of the primitive idempotents such that $E=E_1$. Γ is said to be *Q-polynomial* whenever Γ is Q-polynomial with respect to at least one primitive idempotent. We will use the following characterization of the Q-polynomial property.

Theorem 2.9 (Terwilliger [26, Theorem 3.3(vii)]). *Let $\Gamma=(X,R)$ be a distance-regular graph with diameter d. Let E denote a primitive idempotent of Γ, and let $\theta_0^*,\theta_1^*,\ldots,\theta_d^*$ denote the dual eigenvalues of Γ associated with E. Then Γ is Q-polynomial with respect to E if and only if (i) $\theta_i^*\neq\theta_0^*$ $(1\leqslant i\leqslant d)$, and (ii) for all j,k $(1\leqslant j,k\leqslant d)$ and for all vertices y and y'*

$$\sum_{u\in\Gamma_j(y)\cap\Gamma_k(y')} E\hat{u} - \sum_{v\in\Gamma_k(y)\cap\Gamma_j(y')} E\hat{v}\in \operatorname{span}\{E\hat{y}-E\hat{y}'\}. \tag{2}$$

Lemma 2.10. *Let $\Gamma=(X,R)$ denote a distance-regular graph with diameter $d\geqslant 2$. Let W denote a spin model and assume that Γ supports W. Let $E_0,E_1,\ldots,E_d$ denote the primitive idempotents of Γ in the standard order with respect to W. Then $E_0,E_1,\ldots,E_d$ is a Q-polynomial ordering.*

Proof. The metric property of ∂ implies that for all distinct i and j, $A\circ(A_iA_j)\neq 0$ if and only if $|i-j|\leqslant 1$. Applying Ψ to this relation shows that $E_0,E_1,\ldots,E_d$ is a Q-polynomial ordering. □

Definition 2.11. Let $\Gamma=(X,R)$ denote a distance-regular graph with diameter $d\geqslant 2$. Let W denote a spin model, assume that Γ supports W. An ordering $\theta_0,\theta_1,\ldots,\theta_d$ of the eigenvalues of Γ is said to be *standard with respect to W* if the corresponding ordering $E_0,E_1,\ldots,E_d$ of the primitive idempotents of Γ is standard with respect to W. By the *standard dual eigenvalues with respect to W*, we mean the dual eigenvalues with respect to E_1, where $E_0,E_1,\ldots,E_d$ denote the primitive idempotents of Γ in the standard order with respect to W.

Lemma 2.12. *Let $\Gamma=(X,R)$ denote a distance-regular graph with diameter $d\geqslant 2$. Let W denote a spin model, assume that Γ supports W. Let $\theta_0,\theta_1,\ldots,\theta_d$ denote the eigenvalues of Γ in the standard order with respect to W, and let $\theta_0^*,\theta_1^*,\ldots,\theta_d^*$ denote the standard dual eigenvalues of Γ with respect to W. Then $\theta_i^*=\theta_i$ $(0\leqslant i\leqslant d)$.*

Proof. Applying Ψ to $A=\sum_{i=0}^{d}\theta_i E_i$ shows that $\theta_i^*=\theta_i$ $(0\leqslant i\leqslant d)$. $\square$

We now breifly review the Terwilliger algebra of a distance-regular graph. For more on the Terwilliger algebra see [23–25]. Fix any $x\in X$. For each integer i $(0\leqslant i\leqslant d)$, let $E_i^*=E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\boldsymbol{C})$ with (y,y)-entry $E_i^*(y,y)=1$ if $\partial(x,y)=i$, and 0 otherwise. Let $T=T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\boldsymbol{C})$ generated by $A,E_0^*,E_1^*,\dots,E_d^*$. The algebra T is called the *Terwilliger (or subconstituent) algebra of Γ with respect to x.*

By a *T-module* we mean a subspace $U\subseteq V$ such that $TU\subseteq U$. A T-module U is said to be *irreducible* whenever $U\neq 0$ and U contains no T-modules other than 0 and U. An irreducible T-module U is called *thin* if $\dim E_i^*U\leqslant 1$ $(0\leqslant i\leqslant d)$. The graph Γ is said to be *thin with respect to* x when every irreducible T-module is thin. The graph Γ is said to be *thin* when Γ is thin with respect to every $x\in X$.

For the Q-polynomial distance-regular graphs, the thin property has been characterized by a simple combinatorial condition:

Theorem 2.13 (Terwilliger [25, Theorem 5.1]). *Let $\Gamma=(X,R)$ be a distance-regular graph with diameter d and suppose that Γ is Q-polynomial. Fix $x\in X$. Then Γ is thin with respect to x if and only if for all i $(2\leqslant i\leqslant d-1)$ and for all $w,w'\in\Gamma_i(x)$, $|\Gamma_{i-1}(x)\cap\Gamma_1(w)\cap\Gamma_2(w')|=|\Gamma_{i-1}(x)\cap\Gamma_2(w)\cap\Gamma_1(w')|$.*

We conclude this section by recalling some facts from [12] about distance-regular graphs which support a spin model.

Lemma 2.14 (Curtin and Nomura [12, Lemma 4.2]). *Let $\Gamma=(X,R)$ denote a distance-regular graph with diameter $d\geqslant 2$. Let W denote a spin model, and assume that Γ supports W. Let $t_0,t_1,\dots,t_d$ denote the weights of W with respect to Γ, and let $\theta_0,\theta_1,\dots,\theta_d$ denote the eigenvalues of Γ in the standard order with respect to W. Fix $u,v\in X$, and write $h=\partial(u,v)$. Then for all r,s $(0\leqslant r,s\leqslant d)$ and for all $w\in\Gamma_r(u)\cap\Gamma_s(v)$,*

$$\sum_{i=r-1}^{r+1}\sum_{j=s-1}^{s+1}|\Gamma_1(w)\cap\Gamma_i(u)\cap\Gamma_j(v)|\frac{t_i}{t_j}=\theta_h\frac{t_r}{t_s},\tag{3}$$

$$\sum_{i=r-1}^{r+1}\sum_{j=s-1}^{s+1}|\Gamma_1(w)\cap\Gamma_i(u)\cap\Gamma_j(v)|\frac{t_j}{t_i}=\theta_h\frac{t_s}{t_r}.\tag{4}$$

Lemma 2.15 (Curtin and Nomura [12, Lemma 4.5]). *Let $\Gamma=(X,R)$ denote a distance-regular graph with diameter $d\geqslant 2$. Let W denote a spin model, and assume that Γ supports W. Let $t_0,t_1,\dots,t_d$ denote the weights of W with respect to Γ, and let $\theta_0^*,\theta_1^*,\dots,\theta_d^*$ denote the standard dual eigenvalues of Γ with respect to W. Then for all i $(1\leqslant i\leqslant d)$*

$$\frac{t_i/(t_{i-1})-(t_{i-1})/t_i}{\theta_{i-1}^*-\theta_i^*}=\frac{(t_1/t_0)-(t_0/t_1)}{\theta_0^*-\theta_1^*}.\tag{5}$$

Corollary 2.16 (Curtin and Nomura [12, Corollaries 4.6 and 4.7]). *Let $\Gamma=(X,R)$ denote a distance-regular graph with diameter $d\geqslant 2$. Let W denote a spin model, and assume that Γ supports W. Let $t_0,t_1,\ldots,t_d$ denote the weights of W with respect to Γ.*

(i) *If $t_j\in\{t_{j-1},-t_{j-1}\}$ for some j $(1\leqslant j\leqslant d)$, then $t_i\in\{t_0,-t_0\}$ $(0\leqslant i\leqslant d)$.*

(ii) *If $t_{j-1}=t_{j+1}$ for some j $(1\leqslant j\leqslant d-1)$, then $t_i\in\{t_0,-t_0\}$ $(0\leqslant i\leqslant d)$.*

The spin models which have only two distinct entries have been discussed in a more general context in [15].

3. Main result

Throughout this section $\Gamma=(X,R)$ shall denote a distance-regular graph with diameter $d\geqslant 3$ and W shall denote a spin model. We assume that Γ supports W, and write $t_0,t_1,\ldots,t_d$ to denote the weights of W with respect to Γ. We further assume that the weights $t_0,t_1,\ldots,t_d$ satisfy $t_i\notin\{t_0,-t_0\}$ for all i $(1\leqslant i\leqslant d)$. Throughout this section, $\theta_0,\theta_1,\ldots,\theta_d$ will denote the eigenvalues of Γ in the standard order with respect to W, and $\theta_0^*,\theta_1^*,\ldots,\theta_d^*$ will represent the standard dual eigenvalues of Γ with respect to W.

Definition 3.1. Fix i $(2\leqslant i\leqslant d-1)$, and fix $x,y,y'\in X$ with $\partial(x,y)=i$, $\partial(x,y')=i$, and $\partial(y,y')=3$. For all h $(2\leqslant h\leqslant 4)$ and for all j $(i-1\leqslant j\leqslant i+1)$, define $I_j^h=I_j^h(i,x,y,y')$ by $I_j^h=|\Gamma_j(x)\cap\Gamma_1(y)\cap\Gamma_h(y')|$, and define $I_j'^h=I_j'^h(i,x,y,y')$ by $I_j'^h=|\Gamma_j(x)\cap\Gamma_1(y')\cap\Gamma_h(y)|$.

Observe that by elementary counting arguments $I_{i+1}^3=b_i-I_{i+1}^2-I_{i+1}^4$, $I_{i-1}^3=c_i-I_{i-1}^2-I_{i-1}^4$, $I_i^2=c_3-I_{i+1}^2-I_{i-1}^2$, $I_i^4=b_3-I_{i+1}^4-I_{i-1}^4$, and $I_i^3=a_3-b_i-c_i+I_{i+1}^2+I_{i+1}^4+I_{i-1}^2+I_{i-1}^4$. Thus we will always simplify our expressions so that only the four terms $I_{i-1}^2,I_{i-1}^4,I_{i+1}^2$, and I_{i+1}^4 appear. We will similarly simplify expressions involving the $I_j'^h$.

Lemma 3.2. *With the notation of Definition* 3.1,

$$0=-(I_{i+1}^2-I_{i+1}'^2)(t_{i+1}^2-t_i^2)t_{i-1}+(I_{i-1}^2-I_{i-1}'^2)(t_i^2-t_{i-1}^2)t_{i+1},\tag{6}$$

$$0=-(I_{i+1}^4-I_{i+1}'^4)(t_{i+1}^2-t_i^2)t_{i-1}+(I_{i-1}^4-I_{i-1}'^4)(t_i^2-t_{i-1}^2)t_{i+1}.\tag{7}$$

Proof. To see (6), set $j=1$ and $k=2$ in (2) and take the inner product of both sides with $E\hat{x}$ to obtain

$$0=(I_{i+1}^2-I_{i+1}'^2)(\theta_{i+1}^*-\theta_i^*)+(I_{i-1}^2-I_{i-1}'^2)(\theta_{i-1}^*-\theta_i^*).$$

Eliminating the dual eigenvalues using (5), we routinely obtain (6). Equation (7) is proved similarly by setting $j=1$ and $k=4$ in (2). $\square$

Lemma 3.3. *With the notation of Definition* 3.1,

$$\begin{aligned}0 = &(I^2_{i+1} - I'^2_{i+1})(t_2^{-1} - t_3^{-1})(t_{i+1} - t_i) + (I^4_{i+1} - I'^4_{i+1})(t_4^{-1} - t_3^{-1})(t_{i+1} - t_i)\\&+ (I^2_{i-1} - I'^2_{i-1})(t_2^{-1} - t_3^{-1})(t_{i-1} - t_i)\\&+ (I^4_{i-1} - I'^4_{i-1})(t_4^{-1} - t_3^{-1})(t_{i-1} - t_i),\end{aligned} \tag{8}$$

$$\begin{aligned}0 = &(I^2_{i+1} - I'^2_{i+1})(t_2 - t_3)(t_{i-1}^{-1} - t_i^{-1}) + (I^4_{i+1} - I'^4_{i+1})(t_4 - t_3)(t_{i-1}^{-1} - t_i^{-1})\\&+ (I^2_{i-1} - I'^2_{i-1})(t_2 - t_3)(t_{i-1}^{-1} - t_i^{-1}) + (I^4_{i-1} - I'^4_{i-1})(t_4 - t_3)(t_{i-1}^{-1} - t_i^{-1}).\end{aligned} \tag{9}$$

Proof. To prove (8), first apply (3) with $u = x$, $v = y'$, and $w = y$. After simplifying this gives

$$\begin{aligned}\theta_i\frac{t_i}{t_3} = &I^2_{i+1}\left(\frac{t_{i+1}}{t_2} - \frac{t_{i+1}}{t_3} - \frac{t_i}{t_2} + \frac{t_i}{t_3}\right) + I^4_{i+1}\left(\frac{t_{i+1}}{t_4} - \frac{t_{i+1}}{t_3} - \frac{t_i}{t_4} + \frac{t_i}{t_3}\right)\\&+ I^2_{i-1}\left(\frac{t_{i-1}}{t_2} - \frac{t_i}{t_2} - \frac{t_{i-1}}{t_3} + \frac{t_i}{t_3}\right) + I^4_{i-1}\left(\frac{t_{i-1}}{t_4} - \frac{t_i}{t_4} - \frac{t_{i-1}}{t_3} + \frac{t_i}{t_3}\right)\\&+ b_i\frac{t_{i+1}}{t_3} + c_3\frac{t_i}{t_2} + b_3\frac{t_i}{t_4} + c_i\frac{t_{i-1}}{t_3} + (a_3 - b_i - c_i)\frac{t_i}{t_3}.\end{aligned}$$

Now apply (3) with $u = x$, $v = y$, and $w = y'$. This yields a second equation which are identical to the one which appears immediately above and except that each I^h_j is replaced by I'^h_j. Subtracting this second equation from the first yields (8). Equation (9) is proved similarly using (4) in place of (3). □

Lemma 3.4. *With the notation of Definition* 3.1, $I^h_j = I'^h_j$ $(2 \leqslant h \leqslant 4,\ i-1 \leqslant j \leqslant i+1)$.

Proof. Observe that equations (6), (7), (8), and (9) give a system of four homogeneous linear equations in the four variables $I^2_{i+1} - I'^2_{i-1}, I^4_{i-1} - I'^4_{i-1}, I^2_{i-1} - I'^2_{i-1}$, and $I^4_{i-1} - I'^4_{i-1}$. The coefficient matrix of this system has determinant

$$\frac{(t_3 - t_2)(t_2 - t_4)(t_4 - t_3)(t_i - t_{i-1})^2(t_{i+1} - t_{i-1})^2(t_i - t_{i+1})^2}{t_2t_3t_4}.$$

This determinant is non-zero by Corollary 2.16 since $t_i \notin \{t_0, -t_0\}$ for all i $(1 \leqslant i \leqslant d)$. Hence, all four variables are zero, and the result follows. □

Definition 3.5. Fix i $(2 \leqslant i \leqslant d-1)$, and pick $x, z, z' \in X$ with $\partial(x,z) = i$, $\partial(x,z') = i$, and $\partial(z,z') = 2$. For all h $(1 \leqslant h \leqslant 3)$ and for all j $(i-1 \leqslant j \leqslant i+1)$, define $H^h_j = H^h_j(i,x,z,z')$ by $H^h_j = |\Gamma_j(x) \cap \Gamma_1(z) \cap \Gamma_h(z')|$, and define $H'^h_j = H'^h_j(i,x,z,z')$ by $H'^h_j = |\Gamma_j(x) \cap \Gamma_1(z') \cap \Gamma_h(z)|$.

Observe that by elementary counting arguments $H^2_{i+1} = b_i - H^3_{i+1} - H^1_{i+1}, H^2_{i-1} = c_i - H^3_{i-1} - H^1_{i-1}, H^1_i = c_2 - H^1_{i+1} - H^1_{i-1}, H^3_i = b_2 - H^3_{i+1} - H^3_{i-1}$, and $H^2_i = a_2 - b_i - c_i +$

$H_{i+1}^1 + H_{i+1}^3 + H_{i-1}^1 + H_{i-1}^3$. Similar expressions hold for the $H_j'^h$. Also observe that $H_j^1 = H_j'^1$ for all j $(i-1 \leqslant j \leqslant i+1)$.

Lemma 3.6. *With the notation of Definition* 3.5,

$$0 = (H_{i+1}^3 - H_{i+1}'^3)(t_{i+1} - t_i) + (H_{i-1}^3 - H_{i-1}'^3)(t_{i-1} - t_i), \tag{10}$$

$$0 = (H_{i+1}^3 - H_{i+1}'^3)(t_{i+1}^{-1} - t_i^{-1}) + (H_{i-1}^3 - H_{i-1}'^3)(t_{i-1}^{-1} - t_i^{-1}). \tag{11}$$

Proof. Proceeding as in the proof of Lemma 3.3, we first apply (3) with $u = x$, $v = z'$, and $w = z$, and then we subtract the analogous equation arising when we take $u = x$, $v = z$, and $w = z'$. This gives (10). Equation (11) is proved similarly using (4) in place of (3). □

Lemma 3.7. *With the notation of Definition* 3.5, $H_j^h = H_j'^h$ $(1 \leqslant h \leqslant 3,\ i-1 \leqslant j \leqslant i+1)$.

Proof. Observe that equations (10) and (11) give a system of two homogeneous linear equations in the two variables $H_{i+1}^3 - H_{i+1}'^3$ and $H_{i-1}^3 - H_{i-1}'^3$. The coefficient matrix of this system has determinant

$$\frac{(t_i - t_{i-1})(t_{i+1} - t_{i-1})(t_{i+1} - t_i)}{t_{i-1} t_i t_{i+1}}.$$

This determinant is non-zero by Corollary 2.16 since $t_i \notin \{t_0, -t_0\}$ for all i $(1 \leqslant i \leqslant d)$. Thus both variables are zero, and the result follows. □

Theorem 3.8. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 3$, and let W denote a spin model. Assume that Γ supports W and that the weights $t_0, t_1, \ldots, t_d$ of W with respect to Γ satisfy $t_i \notin \{t_0, -t_0\}$ for all i $(1 \leqslant i \leqslant d)$. Then Γ is thin.*

Proof. Fix i $(2 \leqslant i \leqslant d-1)$, and fix vertices $x, w, w' \in X$ such that $\partial(x, w) = i$, $\partial(x, w') = i$. By Theorem 2.13 it suffices to show that

$$|\Gamma_{i-1}(x) \cap \Gamma_2(w) \cap \Gamma_1(w')| = |\Gamma_{i-1}(x) \cap \Gamma_1(w) \cap \Gamma_2(w')|. \tag{12}$$

Observe that the left-hand side of (12) equals $c_i - |\Gamma_{i-1}(x) \cap \Gamma_1(w) \cap \Gamma_1(w')|$ if $\partial(w, w') = 1$, $H_{i-1}'^2(i, x, w, w')$ if $\partial(w, w') = 2$, $I_{i-1}'^2(i, x, w, w')$ if $\partial(w, w') = 3$, and 0 otherwise. The right-hand side of (12) equals $c_i - |\Gamma_{i-1}(x) \cap \Gamma_1(w) \cap \Gamma_1(w')|$ if $\partial(w, w') = 1$, $H_{i-1}^2(i, x, w, w')$ if $\partial(w, w') = 2$, $I_{i-1}^2(i, x, w, w')$ if $\partial(w, w') = 3$, and 0 otherwise. Now (12) holds in light of Lemmas 3.4 and 3.7, and the result follows. □

Corollary 3.9. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 3$, and let W denote a spin model. Assume that W lies in the Bose–Mesner algebra* *of Γ. If the weights $t_0, t_1, \ldots, t_d$ of W with respect to Γ are distinct from one another and from $-t_0$, then Γ is thin.*

Proof. Clear from Lemma 2.8 and Theorem 3.8. □

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